

2d $\mathcal{N} = (2, 2)$ supersymmetry with $U(1)_{R_V}$ in curved space

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Summary

Based on:

- C. Closset, SC, *to appear*.
- F. Benini, SC, “Partition functions of $\mathcal{N} = (2, 2)$ gauge theories on S^2 and vortices”, arXiv:1206.2356 [hep-th].

What we do:

- Define $\mathcal{N} = (2, 2)$ field theories on compact Riemann surfaces
- Classify supersymmetric backgrounds

Why:

- Exact results by localization
- Type II strings on Calabi-Yau
- 4d $\mathcal{N} = 2$ field theories

- 1 $2d \mathcal{N} = (2, 2)$ SUSY in flat space
- 2 $2d \mathcal{N} = (2, 2)$ SUSY with $U(1)_{R_V}$ in curved space
 - Killing spinor equations
 - Supersymmetry algebra
 - Supermultiplets
 - Supersymmetric Lagrangians
- 3 Supersymmetric backgrounds
- 4 Outlook

2d $\mathcal{N} = (2, 2)$ SUSY in flat space

Superalgebra

$$\{Q_{\pm}, \tilde{Q}_{\pm}\} = -P_{\pm\pm}$$

$$\{Q_{\mp}, \tilde{Q}_{\pm}\} = 0$$

$$\{Q_{-}, Q_{+}\} = \tilde{Z}$$

$$\{\tilde{Q}_{-}, \tilde{Q}_{+}\} = Z$$

$$\{Q_{-}, \tilde{Q}_{+}\} = W$$

$$\{\tilde{Q}_{-}, Q_{+}\} = \tilde{W}$$

$$[M, Q_{\mp}] = \pm \frac{1}{2} Q_{\mp}$$

$$[M, \tilde{Q}_{\mp}] = \pm \frac{1}{2} \tilde{Q}_{\mp}$$

 $U(1)_{R_A} \times U(1)_{R_V}$ automorphism

$$[R_V, Q_{\mp}] = -Q_{\mp}$$

$$[R_V, \tilde{Q}_{\mp}] = +\tilde{Q}_{\mp}$$

$$[R_A, Q_{\mp}] = \pm Q_{\mp}$$

$$[R_A, \tilde{Q}_{\mp}] = \mp \tilde{Q}_{\mp}$$

$$[R_V, Z] = 2Z$$

$$[R_V, \tilde{Z}] = -2\tilde{Z}$$

$$[R_A, W] = 2W$$

$$[R_A, \tilde{W}] = -2\tilde{W}$$

$$[R_V, W] = [R_V, \tilde{W}] = 0$$

$$[R_A, Z] = [R_A, \tilde{Z}] = 0$$

\mathbb{Z}_2 mirror automorphism: $\theta^- \leftrightarrow -\tilde{\theta}^-$

$$Q_- \leftrightarrow \tilde{Q}_-, \quad R_V \leftrightarrow R_A, \quad Z \leftrightarrow W, \quad \tilde{Z} \leftrightarrow \tilde{W}$$

2d $\mathcal{N} = (2, 2)$ on flat \mathbb{R}^2 : supermultiplets

- **Chiral Multiplet** Φ : $\tilde{D}_- \Phi = \tilde{D}_+ \Phi = 0$.

Can have nontrivial R_V , W , \tilde{W} charges.

- **Vector Multiplet**: $\mathcal{V} = \tilde{\mathcal{V}} \sim \mathcal{V} + \Phi + \tilde{\Phi}$.

- **Twisted Chiral Multiplet** Ω : $D_- \Omega = \tilde{D}_+ \Omega = 0$. [Gates, Hull, Roček 1984]

Can have nontrivial R_A , Z , \tilde{Z} charges.

- **Twisted Vector Multiplet**: $\hat{\mathcal{V}} = \tilde{\hat{\mathcal{V}}} \sim \hat{\mathcal{V}} + \Omega + \tilde{\Omega}$.

$Z = \tilde{Z} = 0$, Axial R-symmetry not required.

Flat space superalgebra

$$\{Q_{\pm}, \tilde{Q}_{\pm}\} = -P_{\pm\pm}$$

$$\{Q_{\mp}, \tilde{Q}_{\pm}\} = 0$$

$$\{Q_{-}, Q_{+}\} = 0$$

$$\{\tilde{Q}_{-}, \tilde{Q}_{+}\} = 0$$

$$\{Q_{-}, \tilde{Q}_{+}\} = W$$

$$\{\tilde{Q}_{-}, Q_{+}\} = \tilde{W}$$

$$[M, Q_{\mp}] = \pm \frac{1}{2} Q_{\mp}$$

$$[M, \tilde{Q}_{\mp}] = \pm \frac{1}{2} \tilde{Q}_{\mp}$$

$$[R_V, Q_{\mp}] = -Q_{\mp}$$

$$[R_V, \tilde{Q}_{\mp}] = +\tilde{Q}_{\mp}$$

$$[R_V, W] = 0$$

$$[R_V, \tilde{W}] = 0$$

We will generalize this supersymmetry algebra to curved space
à la [Festuccia, Seiberg 2011].

2d $\mathcal{N} = (2, 2)$ SUSY with $U(1)_{R_V}$
in curved space

Coupling to background supergravity

Couple the 2d \mathcal{R} -multiplet [Dumitrescu, Seiberg 2011] to external new minimal SUGRA multiplet, similarly to [Closset, Dumitrescu, Festuccia, Komargodski 2012] in 3d.

Supercurrent \mathcal{R} -multiplet	$T^{\mu\nu}$	$S^{\mu\alpha}$	$j_{R_V}^\mu$	G_μ	\tilde{G}_μ
New min SUGRA multiplet	$h_{\mu\nu}$	$\psi_{\mu\alpha}$	A_μ	\tilde{C}_μ	C_μ

- $j_{R_V}^\mu$, $-\varepsilon^{\mu\nu}\tilde{G}_\nu$, $\varepsilon^{\mu\nu}G_\nu$: conserved currents for R_V , W , \tilde{W} .
- A_μ , C_μ , \tilde{C}_μ : conjugate gauge fields.

Killing spinor equations

- Frame: $\{e^1 = \sqrt{g}^{\frac{1}{2}} dz, e^{\bar{1}} = \sqrt{g}^{\frac{1}{2}} d\bar{z}\}$
- Metric: $ds^2 = \sqrt{g} dz d\bar{z}$

Killing spinor equations

$$(\nabla_1 - i A_1)\zeta_- = 0 \qquad (\nabla_{\bar{1}} - i A_{\bar{1}})\zeta_- = \frac{1}{2}\mathcal{H}\zeta_+$$

$$(\nabla_1 - i A_1)\zeta_+ = \frac{1}{2}\tilde{\mathcal{H}}\zeta_- \qquad (\nabla_{\bar{1}} - i A_{\bar{1}})\zeta_+ = 0$$

$$(\nabla_1 + i A_1)\tilde{\zeta}_- = 0 \qquad (\nabla_{\bar{1}} + i A_{\bar{1}})\tilde{\zeta}_- = \frac{1}{2}\tilde{\mathcal{H}}\tilde{\zeta}_+$$

$$(\nabla_1 + i A_1)\tilde{\zeta}_+ = \frac{1}{2}\mathcal{H}\tilde{\zeta}_- \qquad (\nabla_{\bar{1}} + i A_{\bar{1}})\tilde{\zeta}_+ = 0$$

where $\tilde{\mathcal{H}} = -i\varepsilon^{\mu\nu}\partial_\mu\tilde{C}_\nu$, $\mathcal{H} = -i\varepsilon^{\mu\nu}\partial_\mu C_\nu$.

Curved space supersymmetry algebra

$$\begin{aligned} \{\delta_\zeta, \delta_{\tilde{\zeta}}\} \varphi_{(r,w,\tilde{w})} &= -2i \left(\mathcal{L}'_K + \zeta_+ \tilde{\zeta}_- \left(w - \frac{r}{2} \mathcal{H} \right) - \zeta_- \tilde{\zeta}_+ \left(\tilde{w} - \frac{r}{2} \tilde{\mathcal{H}} \right) \right) \varphi_{(r,w,\tilde{w})} \\ \{\delta_\zeta, \delta_\eta\} \varphi_{(r,w,\tilde{w})} &= 0 & \{\delta_{\tilde{\zeta}}, \delta_{\tilde{\eta}}\} \varphi_{(r,w,\tilde{w})} &= 0 \end{aligned}$$

where

$$\mathcal{L}'_K \varphi_{(r,w,\tilde{w})} = (K^\mu D_\mu + \frac{i}{2} (D_\mu K_\nu) S^{\mu\nu}) \varphi_{(r,w,\tilde{w})}$$

is the totally covariant Lie derivative along the Killing vector $K^\mu = \zeta \gamma^\mu \tilde{\zeta}$,

$$D_\mu \varphi_{(r,w,\tilde{w})} \equiv \left(\nabla_\mu - ir A_\mu + \frac{1}{2} (w \tilde{C}_\mu - \tilde{w} C_\mu) \right) \varphi_{(r,w,\tilde{w})}$$

the totally covariant derivative of a field $\varphi_{(r,w,\tilde{w})}$.

Supersymmetry transformations

Can be written for a general multiplet. *E.g.*:

Chiral multiplet of charges (r, w, \tilde{w})

$$\delta\phi = \sqrt{2}(\zeta_+\psi_- - \zeta_-\psi_+)$$

$$\delta\psi_- = \sqrt{2}\zeta_-F - \sqrt{2}i\tilde{\zeta}_- \left(w - \frac{r}{2}\mathcal{H} \right) \phi + 2\sqrt{2}i\tilde{\zeta}_+ D_{\bar{1}}\phi$$

$$\delta\psi_+ = \sqrt{2}\zeta_+F - \sqrt{2}i\tilde{\zeta}_+ \left(\tilde{w} - \frac{r}{2}\tilde{\mathcal{H}} \right) \phi + 2\sqrt{2}i\tilde{\zeta}_- D_{\bar{1}}\phi$$

$$\delta F = \sqrt{2}i \left(\tilde{w} - \frac{r-2}{2}\tilde{\mathcal{H}} \right) \tilde{\zeta}_+\psi_- - \sqrt{2}i \left(w - \frac{r-2}{2}\mathcal{H} \right) \tilde{\zeta}_-\psi_+ \\ + 2\sqrt{2}iD_1(\tilde{\zeta}_+\psi_+) - 2\sqrt{2}iD_{\bar{1}}(\tilde{\zeta}_-\psi_-)$$

Supersymmetric Lagrangians

Kinetic Lagrangian

$$\begin{aligned}\mathcal{L}_{\tilde{\phi}\phi} = & 4D_1\tilde{\phi}D_{\bar{1}}\phi - \tilde{F}F + 2i\tilde{\psi}_+D_1\psi_+ - 2i\tilde{\psi}_-D_{\bar{1}}\psi_- + \\ & + \left(w - \frac{r}{2}\mathcal{H}\right) \left(\tilde{w} - \frac{r}{2}\tilde{\mathcal{H}}\right) \tilde{\phi}\phi - \frac{r}{4}R\tilde{\phi}\phi + \frac{1}{2} \left(\tilde{w}\mathcal{H} + w\tilde{\mathcal{H}}\right) \tilde{\phi}\phi + \\ & + i \left(\tilde{w} - \frac{r}{2}\tilde{\mathcal{H}}\right) \tilde{\psi}_+\psi_- - i \left(w - \frac{r}{2}\mathcal{H}\right) \tilde{\psi}_-\psi_+\end{aligned}$$

F-term Lagrangian from superpotential $W(\Phi^i)$ of $r = 2$, $w = \tilde{w} = 0$

$$\mathcal{L}_{F+\tilde{F}} = F^i\partial_i W + \psi_-^i\psi_+^j\partial_i\partial_j W + \tilde{F}^{\bar{i}}\partial_{\bar{i}}\tilde{W} + \tilde{\psi}_+^{\bar{i}}\tilde{\psi}_-^{\bar{j}}\partial_{\bar{i}}\partial_{\bar{j}}\tilde{W}$$

Similarly vector, twisted chiral ([Gomis, Lee 2012] on S^2), twisted vector multiplet.

Supersymmetric backgrounds

Supersymmetric backgrounds: 1 supercharge ζ

R -symmetry line bundle L : $m \equiv c_1(L) = \frac{1}{2\pi} \int_{\Sigma_g} dA = n(g-1) \in \mathbb{Z}$.

R_V -charge 1 Killing spinors

$$\zeta_-(z, \bar{z}) = \lambda s_0 \sqrt{g}^{\frac{1+n}{4}} f_1(\bar{z}), \quad \zeta_+(z, \bar{z}) = \frac{\lambda}{s_0} \sqrt{g}^{\frac{1-n}{4}} f_2(z)$$
$$\overline{f_1(\bar{z})} \in \Gamma(\sqrt{K}^{-1-n}), \quad f_2(z) \in \Gamma(\sqrt{K}^{-1+n})$$

$\overline{f_1(\bar{z})}, f_2(z)$: *regular* holomorphic sections. λ, s_0 : non-vanishing functions.

$U(1)_{R_V}$ connection A

$$A_z = \frac{n}{2} \omega_z - i \partial_z \ln(\lambda s_0), \quad A_{\bar{z}} = \frac{n}{2} \omega_{\bar{z}} - i \partial_{\bar{z}} \ln(\lambda/s_0)$$

$\mathcal{H}, \tilde{\mathcal{H}}$

$$\partial_{\bar{z}} f_1(\bar{z}) + \frac{1+n}{2} f_1(\bar{z}) \partial_{\bar{z}} \ln \sqrt{g} + f_1(\bar{z}) \partial_{\bar{z}} \ln s_0^2 = \frac{1}{2} \mathcal{H} f_2(z) \frac{1}{s_0^2} \sqrt{g}^{\frac{1-n}{2}}$$
$$\partial_z f_2(z) + \frac{1-n}{2} f_2(z) \partial_z \ln \sqrt{g} - f_2(z) \partial_z \ln s_0^2 = \frac{1}{2} \tilde{\mathcal{H}} f_1(\bar{z}) s_0^2 \sqrt{g}^{\frac{1+n}{2}}.$$

Supersymmetric backgrounds: 1 supercharge ζ

$$\deg(\sqrt{K}^{-1\pm n}) = 1 - g \mp m.$$

$g > 1$

- $n = +1$: $(\zeta_-, \zeta_+) = (0, 1)$, $A = +\frac{1}{2}\omega$, $\mathcal{H} = 0$, $\forall \tilde{\mathcal{H}} \rightarrow A$ -twist
- $n = -1$: $(\zeta_-, \zeta_+) = (1, 0)$, $A = -\frac{1}{2}\omega$, $\tilde{\mathcal{H}} = 0$, $\forall \mathcal{H} \rightarrow \bar{A}$ -twist

$g = 1$: T^2

- Trivial bundles: constant f_1, f_2 .

$g = 0$: S^2

- $n = +1$: A -twist, $U(1)$ -equivariant A -twist
- $n = -1$: \bar{A} -twist, $U(1)$ -equivariant \bar{A} -twist
- $n = 0$: $\overline{f_1(\bar{z})}, f_2(z) \in \mathcal{O}(1)$. One zero each.
Can have single supercharge.

Supersymmetric backgrounds: more supercharges

Integrability conditions: ζ

$$2\zeta_- F_{1\bar{1}} + \frac{i}{4}\zeta_- (R - 2\mathcal{H}\tilde{\mathcal{H}}) - i\zeta_+ \partial_{\bar{1}}\mathcal{H} = 0$$

$$2\zeta_+ F_{1\bar{1}} - \frac{i}{4}\zeta_+ (R - 2\mathcal{H}\tilde{\mathcal{H}}) + i\zeta_- \partial_{\bar{1}}\tilde{\mathcal{H}} = 0$$

Integrability conditions: $\tilde{\zeta}$

$$-2\tilde{\zeta}_- F_{1\bar{1}} + \frac{i}{4}\tilde{\zeta}_- (R - 2\mathcal{H}\tilde{\mathcal{H}}) - i\tilde{\zeta}_+ \partial_{\bar{1}}\tilde{\mathcal{H}} = 0$$

$$-2\tilde{\zeta}_+ F_{1\bar{1}} - \frac{i}{4}\tilde{\zeta}_+ (R - 2\mathcal{H}\tilde{\mathcal{H}}) + i\tilde{\zeta}_- \partial_{\bar{1}}\mathcal{H} = 0$$

$F = dA$, R Ricci scalar ($= -2/L^2$ on round S_L^2 of radius L).

2+2 supercharges

$$F = dA = 0, \quad R = 2\mathcal{H}\tilde{\mathcal{H}}, \quad \partial_{\mu}\mathcal{H} = 0, \quad \partial_{\mu}\tilde{\mathcal{H}} = 0$$

- Flat T^2
- Round S_L^2 with $\mathcal{H}\tilde{\mathcal{H}} = -1/L^2$: [Benini, SC 2012; Doroud, Gomis, Le Floch, Lee 2012] and constant complexified $U(1)_{R_A}$ rotations thereof.

Supersymmetric backgrounds: more supercharges

2+0 supercharges

$$F = dA = 0, \quad R = 2\mathcal{H}\tilde{\mathcal{H}}, \quad \partial_z \mathcal{H} = 0, \quad \partial_{\bar{z}} \tilde{\mathcal{H}} = 0$$

1+1 supercharges: Alternatives

- \exists globally defined Killing vector $K = \zeta \gamma^\mu \tilde{\zeta} \partial_\mu$:
 - \mathcal{T}^2
 - $U(1)$ -isometric squashed S^2 :
 - $n = \pm 1$: ($U(1)$ -equivariant) A/\bar{A} -twist
 - $n = 0$: $f_1 = 1, f_2 = iz$ (and conjugate), generalizing [Gomis, Lee 2012].
- No isometries: A/\bar{A} -twist (possibly with $\tilde{\mathcal{H}}$ or $\mathcal{H} \neq 0$).

Outlook

- We have defined 2d $\mathcal{N} = (2, 2)$ field theories with $U(1)_{R_V}$ symmetry in curved space.
- We have classified supersymmetric backgrounds.
- Subclasses of these backgrounds can uplift to 3d or 4d.
- We use these results to study Q -cohomology: whether and how the partition function and correlators depend on various deformations.
- We hope that our study will lead to new exact results in superstring theory, geometry and 4d gauge theories.