Exact Results in Supersymmetric Field Theories
And Applications

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SUSY 2013, Trieste
Introduction

- Wealth of new exact results in supersymmetric gauge theories:
  - Exact computation of novel observables
  - New insights into the non-perturbative dynamics
  - Surprising connections with other areas of physics and maths

- Classic example is the Witten index: $Z(T_D) = \text{Tr} H(-1)^F e^{-\beta H}$

- Order Parameter for SUSY breaking in strongly coupled gauge theories

- Can be computed exactly. It captures $N_{\text{BE}} = 0 - N_{\text{FE}} = 0$

- Gauge theories in curved spacetime yield more refined observables:
  - Probe more deeply the structure of the theory
  - Rich, non-trivial functions of parameters
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**Theme:** Localization of SUSYc QFT’s in non-trivial background geometries
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Theme: **Localization** of SUSY QFT’s in non-trivial **background geometries**

- Classic example is the **Witten index**:

  \[ Z(T^D) = \text{Tr}_\mathcal{H}(-1)^F e^{-\beta H} \]

  - **Order Parameter** for SUSY breaking in strongly coupled gauge theories
  - Can be **computed exactly**. It captures \( N_{E=0}^B - N_{E=0}^F \)

- Gauge theories in **curved spacetime** yield more refined observables:
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Examples

- **Partition function on** $S^1 \times S^{D-1}$  
  - Measures short representations of SUSY, not just the vacuum states
  - Non-trivial function of chemical potentials for charges
  - If theory flows in IR to a SCFT: computes the superconformal index

- **Partition function on** $S^D$
  - In $4d \; \mathcal{N} = 2$ gauge theories it is a non-trivial function of $g^2$
  - $Z(S^D)$ is a fundamental object in QFT:
    - Unified description of c-theorem, a-theorem, F-theorem...
    - Entanglement entropy of CFT with spherical entangling surface in $R^{1,D-1}$

  $$Z(S^D) = e^{-F} \quad F_{UV} > F_{IR}$$
Wealth of new insights into dynamics:

- **S-duality.** Mapping of operators under dualities:
  \[ \langle \text{Wilson loops} \rangle \leftrightarrow \langle \text{'t Hooft loops} \rangle \]
  J.G, Okuda, Pestun

- Dynamics of strongly coupled, IR fixed points of renormalization group:
  - Infrared dualities (Seiberg duality, mirror symm,..) Kapustin, Willet, Yaakov
  - Conformal dimensions of operators at IR fixed points Jafferis

- Surprising connections:
  
  gauge theories \(\leftrightarrow\) 2d nonrational CFT **AGT**

  2d gauge theories \(\leftrightarrow\) exact Kähler potential **Calabi-Yau** compactifications
• Wealth of new insights into dynamics:

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• **Surprising connections:**

  - gauge theories \leftrightarrow 2d nonrational CFT \text{AGT}
  - 2d gauge theories \leftrightarrow exact Kähler potential \text{Calabi-Yau} compactifications

Report on recent advances in **exact results** in the last topic
String Theory Compactifications on Calabi-Yau Manifolds

- Phenomenologically appealing, embedded in a UV complete theory of gravity
- Calabi-Yau NLSM is a 2d $\mathcal{N} = (2, 2)$ SCFT on worldsheet
- Rich interplay between spacetime and worldsheet dynamics

  massless fields $\leftrightarrow$ exactly marginal operators in SCFT
  spacetime couplings $\leftrightarrow$ correlators in SCFT

- Two classes of metric moduli and marginal operators in the SCFT

  complex moduli $\leftrightarrow$ chiral ring: $\mathcal{O}_i \quad i = 1, \ldots, h^{1,2}

  Kähler moduli $\leftrightarrow$ twisted chiral ring: $\mathcal{O}_a \quad a = 1, \ldots, h^{1,1}$

Annihilated by supercharges and R-charges in $\mathcal{N} = (2, 2)$ SCA

$$[q_A, \mathcal{O}_i] = [R, \mathcal{O}_i] = 0$$
$$[q_B, \mathcal{O}_a] = [\mathcal{A}, \mathcal{O}_a] = 0$$
Kinetic energy of moduli captured by the Zamolodchikov metric of SCFT

\[ G_{i\bar{j}}^{K} = \langle \mathcal{O}_i(N) \mathcal{O}_{\bar{j}}(S) \rangle_{S^2} \quad G_{a\bar{b}}^{C} = \langle \mathcal{O}_a(N) \mathcal{O}_{\bar{b}}(S) \rangle_{S^2} \]

Captured by the Kähler potential in Kähler and complex moduli space

\[ G_{i\bar{j}}^{K} = \partial_i \partial_{\bar{j}} \mathcal{K}^K \quad G_{a\bar{b}}^{C} = \partial_a \partial_{\bar{b}} \mathcal{K}^C \]

\[ \mathcal{K}^K \] has nonperturbative, worldsheet instanton corrections. Integrate over

\[ \bar{\partial} \varphi = 0 \]

Worldsheet instanton generating function

\[ F = \sum_{\beta \in H_2(X)} N_{\beta} e^{2\pi i \int_{\beta} t} \]

\[ N_{\beta} : \text{Gromov-Witten invariants} \]

Yukawa couplings also corrected by worldsheet instantons
Strategy

- Realize Calabi-Yau NLSM as gauged linear sigma model (GLSM)

\[ \text{GLSM} \overset{\text{IR}}{\longrightarrow} \text{NLSM} \]

- Represent \( G^K_{ij} \) and \( G^C_{ab} \) as supersymmetric correlators in the GLSM

- Compute exactly by supersymmetric localization of functional integrals
**$\mathcal{N} = (2, 2)$ Supersymmetric Field Theories on $S^2$**

- In the UV, can realize at most an $SU(2|1)$ subalgebra of $\mathcal{N} = (2, 2)$ SCA

\[
\{\bar{Q}_\alpha, Q_\beta\} = \gamma^m_{\alpha\beta} J_m - \frac{1}{2} \varepsilon_{\alpha\beta} T
\]

\(J_m\): $SU(2)$ isometry generators \hspace{1cm} \(T\): R-charge

- There are two inequivalent $SU(2|1)$ subalgebras on the two-sphere:

| $SU(2|1)_A$        | $SU(2|1)_B$        |
|---------------------|---------------------|
| $Q_A$               | $Q_B$               |
| $S_A$               | $S_B$               |
| $R$                 | $A$                 |
| $J_m$               | $J_m$               |

- $\exists$ suitable supercharge $Q_A$ and $Q_B$ in $SU(2|1)_A$ and $SU(2|1)_B$

  $\langle O_i \overline{O}_\bar{j} \rangle$ is $Q_A$-invariant

  $\langle O_a \overline{O}_\bar{b} \rangle$ is $Q_B$-invariant
$\mathcal{N} = (2, 2)$ Supersymmetric Field Theories on $S^2$

- Find a representation of $SU(2|1)_A$ and $SU(2|1)_B$ on supermultiplets

  vector multiplet: $V$    chiral multiplet: $\overline{D}_+ \Phi = \overline{D}_- \Phi = 0$

Additional multiplets are available in $d = 2$. In particular

  twisted chiral multiplet: $\overline{D}_+ Y = D_- Y = 0$

- Lagrangian is a deformation of covariantized flat space one

  \[ \mathcal{L} = \mathcal{L}_0 + \frac{1}{r} \mathcal{L}_1 + \frac{1}{r^2} \mathcal{L}_2 \]

Comments:

- Field content depends on:
  
  $G$: gauge group for vector multiplet
  
  $\mathbf{R}$: representation of $G$ for matter multiplets

- Lagrangian depends on two holomorphic functions:

  $\mathcal{W}(\Phi)$: superpotential    $W(Y)$: twisted superpotential
• Functional integral localizes:

\[ \mathcal{L} \rightarrow \mathcal{L} + t \mathcal{Q} \cdot V \]

\[ \text{path integral independent of } t, \text{ one-loop exact wrt } \hbar_{\text{eff}} = 1/t \]

\[ \text{but exact with respect to original parameters!!} \]

• Correlation function of \( \mathcal{Q} \)-invariant operators given by:

\[ \langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle = \int_{\mathcal{F}} \mathcal{O}_1 \ldots \mathcal{O}_n |\mathcal{F} \rangle Z_{1\text{-loop}} |\mathcal{F} \rangle e^{-S}|\mathcal{F} \rangle \]

\[ \mathcal{F} = \{ \text{Saddle points of } \mathcal{Q} \cdot V: \text{Solution to PDE’s} \} \]
Gauge Theory and Calabi-Yau Kähler potential

- \exists supercharge $Q_A$ and $Q_B$ in $SU(2|1)_A$ and $SU(2|1)_B$ obeying

$$Q_A^2 = J + \frac{R}{2} \quad Q_B^2 = J + \frac{A}{2}$$

such that

$$\langle O_i \overline{O}_j \rangle \text{ is } Q_A\text{-invariant}$$

$$\langle O_a \overline{O}_b \rangle \text{ is } Q_B\text{-invariant}$$

- Correlators in GLSM are independent of gauge couplings
- Gauge theory computes the theory in extreme IR: Calabi-Yau NLSM
- Calabi-Yau moduli appear as parameters in the gauge theory
  - complex moduli: superpotential $W$
  - Kähler moduli: twisted superpotential $W$

$$Z_A = e^{-\kappa^K}, \quad Z_B = e^{-\kappa^C}$$
Field Theory Approach to Kähler potential for Kähler moduli

- Field Theory realizes $SU(2|1)_A$ symmetry

1) Landau-Ginzburg model with $W$. The exact $S^2$ partition function is:

$$Z_A = \int dY d\bar{Y} e^{-4\pi irW(Y) - 4\pi ir\bar{W}(\bar{Y})}$$

2) Gauge Theory can be localized in two complementary ways:

Coulomb phase:

$$Z_C = \sum_B \int_t da e^{-4\pi ir\xi a + i\vartheta B} \prod_{\alpha \in \Delta} \left[ \left( \frac{\alpha \cdot B}{2r} \right)^2 + (\alpha \cdot a)^2 \right] \prod_{w \in \mathbb{R}} \frac{\Gamma \left( -ir(w \cdot a) - \frac{w \cdot B}{2} \right)}{\Gamma \left( 1 + ir(w \cdot a) - \frac{w \cdot B}{2} \right)}$$

Comments:
- Mellin-Barnes representation
- $Z$ factorizes into sum of more fundamental building blocks
- $Z$ annihilated by system of differential operators
Higgs phase:

- $F_{\text{Higgs}}$ are localized vortices and anti-vortices at the poles

\[ \begin{align*}
(D_1 + iD_2)\phi = 0, & \quad (\sigma + m)\phi = 0 \\
F_{12} + iD = 0, & \quad [\sigma, \bar{\sigma}] = 0
\end{align*} \]

- Putting vortices and anti-vortices together we obtain

\[ Z_H = \sum_{v \in \text{vacua}} e^{-S_0} \left| \text{res}_{a=v} \left[ Z_{\text{one-loop}}(a, 0, m) \right] \right| Z_{\text{vortex}}(v, m, e^{2\pi i \tau}) \right|^2 \]

Comments:

- Physics approach to Gromov-Witten invariants
  Jockers, Kumar, Lapan, Morrison, Romo & J.G, Lee

- Applies to Calabi-Yau’s for which no other methods are available

- Does so without using mirror symmetry
Field Theory Approach to Kähler potential for complex moduli

- Field Theory realizes $SU(2|1)_B$ symmetry
- Partition function can be computed by supersymmetric localization
- Gauge theory realization of Kähler potential in complex structure moduli space of a Calabi-Yau $M$

$$Z_B = e^{-\mathcal{K}^C} = \int_M \Omega \wedge \overline{\Omega}$$
Conclusions

- Our understanding of the dynamics in supersymmetric gauge theories has enjoyed significant advances in recent years
  - Realization of ubiquity of new observables in gauge theories
  - Exact computations as a tool into the dynamics
  - Deepened our understanding of dualities
  - Applications to other areas of inquiry

- Other complementary approaches to supersymmetric gauge dynamics
  - Integrability
  - Gauge/Gravity Duality