Supersymmetry beyond Flat Space

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Introduction

An extremely fruitful area of research in the last couple of decades has been the analysis of supersymmetric quantum field theories.

This has often had contact with phenomenology, for example, through dynamical supersymmetry breaking and gauge mediation, or through mechanisms for generating the μ, B_μ-terms.

Introduction

Supersymmetry in flat space leads to detailed understanding of the dynamics of many strongly coupled gauge theories.

Some powerful methods for analyzing such theories have been

- Chiral operators, chiral ring (i.e. operators annihilated by two supercharges).
- General constraints on QFT, such as anomaly matching.

Introduction

In recent years it has been realized that there is a lot more to be uncovered. Very roughly speaking, it turns out that operators that are annihilated by one supercharge are also protected against quantum corrections if counted with signs.

Introduction

The best way to organize this counting problem is to compactify $\mathcal{N} = 1$ supersymmetric theories on $\mathbb{S}^3 \times \mathbb{S}^1$. The partition function is calculable non-perturbatively. It leads to powerful checks of dualities and other dynamical properties of QFT.

Similarly, three-dimensional $\mathcal{N} = 2$ theories on $\mathbb{S}^3$ have also led to a proliferation of new exact results and insights regarding renormalization group flows, entanglement entropy, etc.

[Kinney-Maldacena-Minwalla-Raju, Römelsberger, Pestun (for $\mathcal{N} = 2$), Kapustin-Willet-Yaakov, Jafferis,...]

Some Questions

This suggests that the power of supersymmetry is far from having been exhausted.

- Which other constraints on dynamics can we get?
- Can we study spaces other than $\mathbb{S}^3 \times \mathbb{S}^1$ and $\mathbb{S}^3$?
- If so, it would be nice to understand what the partition function $Z_{\mathcal{M}_d}$ computes and what it depends on.
- Many curious properties of the partition function were observed in the cases $\mathbb{S}^3$ and $\mathbb{S}^3 \times \mathbb{S}^1$ (holomorphy, decomposition into blocks, $SL(3,\mathbb{Z})$....[Jafferis, Pasquetti, Spiridonov-Vartanov,....]) and it is likely that to understand them we need to embed these special cases in a general picture.

Introduction

Our goal is to understand four-dimensional $\mathcal{N} = 1$ theories on four-manifolds $\mathcal{M}_4$.

We will also discuss three-dimensional $\mathcal{N} = 2$ theories on three manifolds $\mathcal{M}_3$.

Introduction

Suppose we start from some generic QFT and we want to couple it to some curved space. We basically replace $\partial_\mu \to \nabla_\mu$ and we worry a little more about fermions.

But the process is not unique.

- We can add couplings to curvature.
- We can turn on additional background fields, such as various gauge fields that couple to currents etc.

Introduction

If we take a supersymmetric theory and just naively replace $\partial_\mu \to \nabla_\mu$ we will have no leftover supersymmetry in curved space.

To preserve some of the flat space supersymmetry in curved space, we need to turn on additional background fields and possibly add non-minimal couplings to gravity.

The R-multiplet

The ambiguities in coupling the theory to curved space can be dealt with systematically.

We start from $\mathcal{N}=1$ in $\mathbb{R}^4$. The energy-momentum tensor resides in a multiplet with bosonic components

$$\left(j_\mu^R, T_{\mu\nu}, F_{\mu\nu}\right) \ ,$$

where $\partial^\mu j_\mu^R=0$, $dF=0$, and the energy-momentum tensor is symmetric and conserved. This is a 12+12 multiplet.

[...,ZK-Seiberg, ...]

The R-multiplet

We couple the R-multiplet to background fields:

$T_{\mu\nu}$	$g_{\mu\nu}$
j_μ^R	A_μ^R
$F_{\mu\nu}$	$\epsilon_{\mu\nu\rho\sigma}B^{\rho\sigma}$

We emphasize that the fields $g_{\mu\nu}, A_\mu^R, B_{\mu\nu}$ are not path integrated over.

For a general choice of these background fields, the coupling to curved space breaks supersymmetry.

The R-multiplet

It turns out that a necessary and sufficient condition to preserve at least one supercharge is that $\mathcal{M}_4$ is complex and the metric is Hermitian:

$$ds^2 = g_{i\bar{j}} dz^i d\bar{z}^{\bar{j}}$$

The algebra satisfied by the single supercharge is $\delta^2 = 0$.

Another curious case is when the manifold is complex and looks locally like $\mathbb{T}^2 \times \Sigma$ where Σ is a Riemann surface. Then, one can preserve two supercharges.

From this point of view, $\mathbb{S}^3 \times \mathbb{S}^1$ is a (Hopf) fibration of $\mathbb{T}^2$ over $\mathbb{S}^2$ and thus clearly preserves two supercharges, but actually, this (and only this) special fibration preserves 4 supercharges.

[Festuccia-Seiberg, Dumitrescu-Festuccia-Seiberg, Klare-Tomasiello-Zaffaroni]

Global Symmetries

Suppose we consider a theory that has a global symmetry group G. The bosonic operators in the supersymmetric current multiplet are

$$(J^G, j^G_\mu) \ .$$

We can couple them to background fields,

j^G_μ	A^G_μ
J^G	D^G

Supersymmetry is broken unless the connection A^G_μ is such that the G-bundle is holomorphic.

Global Symmetries

Hence, we have a leftover supersymmetry if and only if the manifold is complex and the global symmetry G-bundle is holomorphic.

The Partition Function

Let us specify a four-manifold $\mathcal{M}_4$ with some complex structure $J^2 = -1$ and a Hermitian metric $g_{i\bar{j}}$. If there are global symmetries, we can also specify holomorphic G-bundles. So we have

$$Z_{\mathcal{M}_4}[J_i^j, \overline{J}_{\bar{i}}^{\bar{j}}, g_{i\bar{j}}, A_\mu^G, \ldots]$$

The $\cdots$ stand for several additional parameters which enter the partition function, including the parameters of the Lagrangian itself.

The Partition Function

Our main concern is to understand what the partition function depends on, including the parameters parameterizing the geometry and the Lagrangian itself.

The Partition Function

Several general properties of the partition function $Z_{\mathcal{M}_4}[J_i^j, \overline{J}_{\bar{i}}^{\bar{j}}, g_{i\bar{j}}, A_\mu^G, \ldots]$:

- Given the complex structure $J^2 = -1$, the partition function is independent of the Hermitian metric $g_{i\bar{j}}$.
- The dependence on the complex structure moduli is holomorphic.
- The partition function depends holomorphically on the moduli of the holomorphic G-bundle.

The Partition Function

Therefore, the partition function of $\mathcal{N} = 1$ theories computes invariants of the complex structure and of the complex structure of G-bundles.

The Partition Function

Example: $\mathbb{S}^3 \times \mathbb{S}^1$ was studied by Kodaira. Its moduli space of complex structure is two complex dimensional, with two natural coordinates introduced by Kodaira, s, t.

On the other hand, physicists [Römelsberger...] computed the partition function of $\mathcal{N} = 1$ theories on $\mathbb{S}^3 \times \mathbb{S}^1$ and it was found that one can introduce two chemical potentials along the $\mathbb{S}^1$, denoted p, q. Those are in one-to-one correspondence with s, t of Kodaira, and *the correspondence can be established explicitly*.

If we have a $U(1)$ global symmetry, the chemical potential z along the $\mathbb{S}^1$ corresponds to the modulus of a holomorphic $U(1)$ bundle over $\mathbb{S}^3 \times \mathbb{S}^1$.

The Partition Function

The partition function on $\mathbb{S}^3 \times \mathbb{S}^1$ results in various special functions that have received a lot of attention recently [Spiridonov, Rains, van de Bult...]. We have therefore found that they should be interpreted as functions on the moduli space of complex structures of $\mathbb{S}^3 \times \mathbb{S}^1$.

Three-Dimensional $\mathcal{N} = 2$ Theories

One can similarly consider three-dimensional theories with four supercharges on $\mathbb{R}^3$ with an R-symmetry. One can ask which three-manifolds $\mathcal{M}_3$ are consistent with having a leftover supersymmetry.

The technical answer is as follows: Supersymmetry is preserved on almost contact manifolds $\mathcal{M}_3$ which obey an additional integrability condition. In detail, almost contact manifolds have $\eta_\mu, \xi^\mu, \Phi^\mu_\nu$ satisfying

$$\eta_\mu \xi^\mu = 1 \ , \qquad \Phi^\mu{}_\rho \Phi^\rho{}_\nu = -\delta^\mu_\nu + \xi^\mu \eta_\nu \ .$$

and the integrability condition reads

$$\Phi^\mu{}_\nu \mathcal{L}_\eta \Phi^\nu{}_\rho = 0 \ .$$

[Klare-Tomasiello-Zaffaroni, Closset-Dumitrescu-Festuccia-ZK]

Three-Dimensional $\mathcal{N} = 2$ Theories

It appears that this integrability condition was not studied in the mathematical literature. Once we impose it on almost contact three-manifolds, one finds properties tantalizingly similar to complex geometry. There is an analog of Dolbeault cohomology and of Kodaira-Spencer theory. We analyzed basic aspects of the moduli space of integrable almost contact structure on $\mathbb{S}^3$.

We do not have a detailed understanding of which three-manifolds admit this new structure. It definitely exists on Seifert manifolds.

Three-Dimensional $\mathcal{N} = 2$ Theories

The supersymmetric partition function on three-manifolds is only sensitive to the integrable almost contact structure and not to the metric.

On manifolds with the differentiable structure of $\mathbb{S}^3$ there is a one-dimensional space of integrable almost contact structure. This is in correspondence with the squashing parameter denoted b in the literature.

Three-Dimensional $\mathcal{N} = 2$ Theories

Loosely speaking, no matter which new squashing of $\mathbb{S}^3$ is found, there will be no new partition functions beyond the one of Hama-Hosomichi-Lee. This is because the moduli space of integrable almost contact structures is one dimensional.

Actually, Hama-Hosomichi-Lee had two types of squashings, and our result clearly explains why one of them is trivial. (It has the same integrable almost contact structure as the round $\mathbb{S}^3$.)

Three-Dimensional $\mathcal{N} = 2$ Theories

Incidentally, this also explains why Imamura found the same answer as Hama-Hosomichi-Lee. More recently, new compactifications by [Martelli-Passias] appeared. From the considerations above, we predict they would yield the same partition functions as those of Hama-Hosomichi-Lee. This proposal is not inconsistent with their holographic results.

It is also nicely consistent with recent work by [Alday-Martelli-Richmond-Sparks].

Three-Dimensional $\mathcal{N} = 2$ Theories

The dependence on the integrable almost contact structure may provide exact results about non-chiral correlation functions in flat space. For example in [Closset, Dumitrescu, ZK, Seiberg] we have shown how to compute various non-chiral two-point functions in $\mathbb{R}^3$ and used this as tests of dualities and RG flows

Can these results be derived in flat space directly?

Three-Dimensional $\mathcal{N} = 2$ Theories

Note that although the mathematical structure we encounter in 3d seems unfamiliar, it has several interesting properties, and it arises from physics as naturally as complex geometry in 4d.

Conclusions and Open Questions

- We used the tools of generating functionals and studied the partition function as a function of the topology, complex structure, and metric (and several other technical ingredients).
- We found that $4D$ $\mathcal{N}=1$ theories compute invariants of the complex structure, i.e. the partition function depends only on the underlying complex structure.
- $3D$ $\mathcal{N}=2$ theories compute invariants of a certain integrable almost contact structure.
- We discussed a new interpretation for the parameters appearing in the $\mathbb{S}^3\times\mathbb{S}^1$ partition function. We also explained why the partition functions on various squashed three-spheres cannot have more information than the original formula of Hama-Hosomichi-Lee.

Conclusions and Open Questions

- The partition functions as functions of the complex structure moduli can have various poles. What is their meaning?
- Is there an obstruction to have an integrable almost contact structure on three-manifolds?
- When we have meromorphic functions (in this case of the complex structure moduli) we can often fix them by knowing something about the singularities and perhaps about global properties of the moduli space. Can we use these ideas to compute the partition function on any complex $\mathcal{M}_4$?
- Generalize to 6d, 5d, $\mathcal{N} = 2$ in 4d, etc.

Bonus Slides

Thanks for your Attention.