New superspace techniques for conformal supergravity in three dimensions

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Based on:
Kuzenko & GTM, JHEP **1303**, 113 (2013), 1212.6852
Butter, Kuzenko, Novak & GTM, *accepted in JHEP*, 1305.3132
Butter, Kuzenko, Novak & GTM *accepted in JHEP*, 1306.1205
Kuzenko, Novak & GTM arXiv:1308.5552

Outline

Introduction: AIM

Construct the off-shell action
for $\mathcal{N}$-extended conformal supergravity
in three space-time dimensions (3D)

Introduction: old results in 3D conformal SUGRA

in a series of papers between 1985 and 1993 the on-shell 3D $\mathcal{N}$-extended conformal supergravity actions were constructed:

- $\mathcal{N} = 1$:[van Nieuwenhuizen (1985)];
- $\mathcal{N} = 2$:[Roček, van Nieuwenhuizen (1986)];
- general-$\mathcal{N}$: [Lindström, Roček (1989)] & [Nishino, Gates (1993)].

$\mathcal{N} = 1, 2$ actions are off-shell superconformal (no auxiliary fields); while on-shell for $\mathcal{N} > 3$

Introduction: old results in 3D conformal SUGRA, action

- The on-shell action for general $\mathcal{N}$ is a Chern-Simons (CS) type action with Higher Derivatives in gravitini

$$
\begin{aligned}
S_{\rm CS} = \frac{1}{4}\int \mathrm{d}^3x\, e\, \varepsilon^{abc}\Big(&\omega_a{}^{fg}\mathcal{R}_{bc\,fg} - \frac{2}{3}\omega_{af}{}^{g}\omega_{bg}{}^{h}\omega_{ch}{}^{f} \\
&-2\mathcal{R}_{ab}{}^{IJ}V_{c\,IJ} - \frac{4}{3}V_a{}^{IJ}V_{bI}{}^{K}V_{c\,KJ} \\
&-\frac{\mathrm{i}}{2}\Psi_{bc}{}_I^{\alpha}(\gamma_d)_\alpha{}^\beta(\gamma_a)_\beta{}^\gamma\varepsilon^{def}\Psi_{ef}{}_\gamma^I\Big)
\end{aligned}
$$

$\omega_a{}^{bc}$, $\mathcal{R}_{ab}{}^{cd}$ Lorentz connection and curvature,
$V_a{}^{IJ}$, $\mathcal{R}_{ab}{}^{IJ}$ SO($\mathcal{N}$) R-symmetry connection and curvature,
$\Psi_{bc}{}_I^{\alpha}$ gravitini field strength.

- Action was constructed in components as a CS action for the $\mathcal{N}$-extended superconformal algebra $\mathfrak{osp}(\mathcal{N}|4,\mathbb{R})$

How to extend the action off-shell?

Motivations: Why off-shell 3D conformal SUGRA?

An off-shell approach to conformal SUGRA, when available, can be used to generate general supergravity matter couplings:
- superconformal tensor calculus;
- superspace approaches.

Example of recent applications in 3D have seen:

- 3D massive gravity and AdS_3/CFT_2. Topological-Massive-Gravity (TMG) [Li-Song-Strominger ('08)]; New-Massive-Gravity (NMG), Generalized-Massive-Gravity (GMG) [Bergshoeff-Hohm-Townsend ('09)]
- Supersymmetric extensions of NMG & GMG:
  $\mathcal{N} = 1$ non-linear case [Andringa-Bergshoeff-deRoo-Hohm-Sezgin-Townsend ('09)], [Bergshoeff-Hohm-Rosseel-Sezgin-Townsend ('10)]
  $\mathcal{N} \geq 2$ understood at the linearized level [Bergshoeff-Hohm-Rosseel-Townsend ('10)]
- Fully non-linear SUSY difficult due to higher-derivative terms
- TMG, GMG includes conformal SUGRA action as building block

Motivations: Why off-shell 3D conformal SUGRA?

- recent renewed interest in constructing off-shell SUSY theories on constant curvatures backgrounds in 4D and 3D. This opened a classification problem for SUSY backgrounds of off-shell sugra
  see Komargodski's and Tomasiello's talk for list of references
- localization techniques for computing Wilson loop, partition functions, indices... see Gomis's talk for list of references
  Localization needs off-shell SUSY
- in 3D, off-shell curved (Lorentz)-Chern-Simons (conformal sugra actions) have a role in explaining features like contact terms, complexity and F-maximization of the partition function of $\mathcal{N} = 2$ theories [Closset-Dumitrescu-Festuccia-Komargodski-Seiberg ('12)]

Motivations: off-shell 3D superspace SUGRA program

Superspace formulations for 3D $\mathcal{N}$-extended SUGRA help.
Surprisingly this was *not* fully developed in the past

- $\mathcal{N} = 1$: [Howe-Tucker (1977)];[Brown-Gates (1979)]
- $\mathcal{N} \geq 2$ sketched [Howe-Izquierdo-Papadopoulos-Townsend (1995)]
  more for $\mathcal{N} = 8$ [Howe-Sezgin ('04)]

$\Downarrow$

SO($\mathcal{N}$) supergeometry for off-shell Weyl multiplet of conformal SUGRA
for any $\mathcal{N}$ in [Kuzenko-Lindström-GTM ('11)]
and general supergravity-matter systems with $\mathcal{N} \leq 4$

independently:

- $\mathcal{N} = 8$: [Cederwall-Gran-Nilsson ('11)]:
- $\mathcal{N} = 16$: [Greitz-Howe ('11)]

Motivations: off-shell 3D superspace SUGRA program

A missing element in the program was indeed to build the action for $\mathcal{N}$-extended conformal sugra using superspace
(known only for $\mathcal{N} = 1$ [Gates-Grisaru-Roček-Siegel (1981)]; [Zupnik-Pak (1988)])

$\Longrightarrow$ How to build it for general $\mathcal{N}$?

Motivations: How conformal SUGRA action for general $\mathcal{N}$?

- A way is to use superform/ectoplasm techniques to engineer invariant actions from closed super 3-forms.
  Idea similar to action for linear multiplet in 4D $\mathcal{N} = 2$ supergravity [Butter-Kuzenko-Novak ('12)]
- "Ectoplasm" in physics has been rediscovered various time: [Hasler (1996)]; [Gates-Grisaru-Knutt–Wehlau-Siegel (1997)]; see also reheonomy approach (book [Castellani-D'Auria-Fre (1991)]); then systematically developed and used to classify counterterms [Boussard-Howe-Stelle-... ('09)–('13)]

Successful way: [Butter-Kuzenko-Novak-GTM ('13)]

- manifestly gauge entire $\mathfrak{osp}(\mathcal{N}|4, \mathbb{R})$ in superspace
  $\Longrightarrow$ "3D conformal superspace" (turns out to simplify calculations)
- ectoplasm for $\mathcal{N} = 1, \cdots, 5$ [1306.1205]; then $\mathcal{N} = 6$ [1308.5552]

$\mathcal{N} = 3, 4, 5, 6$ actions were never constructed before

(note [Nishimura-Tanii]: $\mathcal{N} = 6$, with tensor calculus, just one week before)

3D conformal supergravity in conformal superspace

- Supergeometry based on a manifest gauging of the entire superconformal group $\mathsf{OSp}(\mathcal{N}|4,\mathbb{R})$
- 3D generalization of the pioneering works by [Butter ('09),('11)] for $\mathcal{N}=1,2$ conformal supergravity
- interpreted as Superspace analogue of superconformal tensor calculus

3D conformal supergravity in conformal superspace

Take an $\mathcal{N}$-extended curved superspace

$$z^M = (x^m, \theta^\mu_{\text{I}}) , \qquad m = 0, 1, 2 , \quad \mu = 1, 2 , \quad \text{I} = \mathbf{1}, \cdots , \boldsymbol{\mathcal{N}}$$

Structure group X is chosen to be:
$\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SO}(\mathcal{N}) \times$(Dilatations)$\times$($S-$susy) $\times$ ($K-$boosts).
The superspace covariant derivatives

$$\nabla_A = E_A{}^M \partial_M - \omega_A{}^{\underline{b}} X_{\underline{b}} = E_A{}^M \partial_M - \frac{1}{2}\Omega_A{}^{ab} M_{ab} - \frac{1}{2}\Phi_A{}^{PQ} N_{PQ} - B_A \mathbb{D} - \mathfrak{F}_A{}^B K_B$$

- $E_A{}^M(z)$ supervielbein, $\partial_M = \partial/\partial z^M$, - $\Omega_A{}^{cd}(z)$ Lorentz connection,
- $\Phi_A(z)$ $\mathrm{SO}(\mathcal{N})$-connection, - B_A dilatation $\mathbb{D}$-connection
- $\mathfrak{F}_A{}^B$ special superconformal connection, $K_A = (K_a, S^I_\alpha)$
- ∇_A gauge super-Poincaré via superdiffeomorphism, $P_A = (P_a, Q^I_\alpha)$.

- The sugra local gauge transformations $\mathrm{OSp}(\mathcal{N}|4, \mathbb{R})$

$$\mathcal{K} := \xi^A \nabla_A + \frac{1}{2}\Lambda^{bc} M_{bc} + \frac{1}{2}\Lambda^{KL} N_{KL} + \sigma \mathbb{D} + \Lambda^A K_A , \qquad \delta_{\mathcal{K}} \nabla_A = [\mathcal{K}, \nabla_A]$$

3D conformal supergravity in conformal superspace

Can prove that the algebra:

$$[\nabla_A, \nabla_B\} = -T_{AB}{}^C \nabla_C - \frac{1}{2} R(M)_{AB}{}^{cd} M_{cd} - \frac{1}{2} R(N)_{AB}{}^{PQ} N_{PQ} - R(\mathbb{D})_{AB} \mathbb{D} - R(S)_{AB}{}^{\gamma}_{K} S^K_{\gamma} - R(K)_{AB}{}^c K_c$$

can be constrained to be for $\mathcal{N} > 3$

$$\{\nabla^I_\alpha, \nabla^J_\beta\} = 2\mathrm{i}\delta^{IJ}\nabla_{\alpha\beta} + \mathrm{i}\varepsilon_{\alpha\beta}W^{IJKL}N_{KL} - \frac{\mathrm{i}}{\mathcal{N}-3}\varepsilon_{\alpha\beta}(\nabla^\gamma_K W^{IJKL})S_{\gamma L} + \frac{1}{2(\mathcal{N}-2)(\mathcal{N}-3)}\varepsilon_{\alpha\beta}(\gamma^c)^{\gamma\delta}(\nabla_{\gamma K}\nabla_{\delta L}W^{IJKL})K_c ,$$

$$[\nabla_a, \nabla^J_\beta] = \frac{1}{2(\mathcal{N}-3)}(\gamma_a)_{\beta\gamma}(\nabla^\gamma_K W^{JPQK})N_{PQ} - \frac{1}{2(\mathcal{N}-2)(\mathcal{N}-3)}(\gamma_a)_{\beta\gamma}(\nabla^\gamma_L\nabla^\delta_P W^{JKLP})S_{\delta K} - \frac{\mathrm{i}}{4(\mathcal{N}-1)(\mathcal{N}-2)(\mathcal{N}-3)}(\gamma_a)_{\beta\gamma}(\gamma^c)_{\delta\rho}(\nabla^\gamma_K\nabla^\delta_L\nabla^\rho_P W^{JKLP})K_c ,$$

$$[\nabla_a, \nabla_b] = \frac{1}{4(\mathcal{N}-2)(\mathcal{N}-3)}\varepsilon_{abc}(\gamma^c)_{\alpha\beta}\Big(\mathrm{i}(\nabla^\alpha_I\nabla^\beta_J W^{PQIJ})N_{PQ} + \frac{\mathrm{i}}{\mathcal{N}-1}(\nabla^\alpha_I\nabla^\beta_J\nabla^\gamma_K W^{LIJK})S_{\gamma L} + \frac{1}{2\mathcal{N}(\mathcal{N}-1)}(\gamma^d)_{\gamma\delta}(\nabla^\alpha_I\nabla^\beta_J\nabla^\gamma_K\nabla^\delta_L W^{IJKL})K_d\Big) ,$$

Algebra parametrized only in terms of one single superfield:
$W^{IJKL} = W^{[IJKL]}$: the super-Cotton tensor
describing 3D $\mathcal{N}$-extended Weyl multiplet

3D conformal supergravity in conformal superspace

W^{IJKL} is a dimension-1 primary

$$S^P_\alpha W^{IJKL} = 0 \ , \quad \mathbb{D} W^{IJKL} = W^{IJKL}$$

and satisfies one simple Bianchi identity (for $\mathcal{N} > 4$)

$$\nabla^I_\alpha W^{JKLP} = \nabla^{[I}_\alpha W^{JKLP]} - \frac{4}{\mathcal{N} - 3} \nabla_{\alpha Q} W^{Q[JKL} \delta^{P]I}$$

Ectoplasms: generalities

- closed super p-form $J = E^{\underline{A}_p} \wedge \cdots \wedge E^{\underline{A}_1} J_{\underline{A}_1 \cdots \underline{A}_p}$, $(E^{\underline{A}} = \mathrm{d}z^{\underline{M}} E_{\underline{M}}{}^{\underline{A}})$

$$0 = (\mathrm{d}J)_{\underline{A}_1 \cdots \underline{A}_p \underline{A}_{p+1}} = \frac{1}{p!} \nabla_{[\underline{A}_1} J_{\underline{A}_2 \cdots \underline{A}_{p+1})} - \frac{1}{2((p-1)!)} T_{[\underline{A}_1 \underline{A}_2|}{}^{\underline{B}} J_{\underline{B}|\underline{A}_3 \cdots \underline{A}_{p+1})}$$

- if $p = d$, dimension of the bosonic body of the superspace, then

$$S = \int_{\mathcal{M}^d} J = \int \mathrm{d}^d x \, {}^*J|_{\theta=0} \ , \qquad {}^*J = \frac{1}{d!} \varepsilon^{\underline{m}_1 \cdots \underline{m}_d} J_{\underline{m}_1 \cdots \underline{m}_d}$$

S invariant under superdiffeomorphism $\xi = \xi^{\underline{A}} E_{\underline{A}} = \xi^{\underline{M}} \partial_{\underline{M}}$,

$$\delta_\xi J = \mathcal{L}_\xi J \equiv i_\xi \mathrm{d}J + \mathrm{d} i_\xi J = \mathrm{d} i_\xi J \ .$$

- By moving to flat indices the action is equivalently written as

$$S = \int \mathrm{d}^d x \frac{1}{d!} \varepsilon^{\underline{m}_1 \cdots \underline{m}_d} E_{\underline{m}_d}{}^{\underline{A}_d} \cdots E_{\underline{m}_1}{}^{\underline{A}_1} J_{\underline{A}_1 \cdots \underline{A}_d}|_{\theta=0} \ ,$$

$$S = \int \mathrm{d}^d x \, \mathrm{e} \, \varepsilon^{\underline{a}_1 \cdots \underline{a}_d} \Big[J_{\underline{a}_1 \cdots \underline{a}_d} + c_2 \Psi_{\underline{a}_1}{}^{\underline{\alpha}_1} J_{\underline{\alpha}_1 \underline{a}_2 \cdots \underline{a}_d} + c_3 \Psi_{\underline{a}_1}{}^{\underline{\alpha}_1} \Psi_{\underline{a}_2}{}^{\underline{\alpha}_2} J_{\underline{\alpha}_1 \underline{\alpha}_2 \underline{a}_3 \cdots \underline{a}_d} + \cdots + c_d \Psi_{\underline{a}_1}{}^{\underline{\alpha}_1} \cdots \Psi_{\underline{a}_d}{}^{\underline{\alpha}_d} J_{\underline{\alpha}_1 \cdots \underline{\alpha}_d} \Big]\Big|_{\theta=0} \ ,$$

Ectoplasms: generalities

- To have an action invariant under the full supergravity gauge group one needs also invariance under the tangent space structure group X
- we need closed forms such that

$$\delta_X J = \mathrm{d}Y(X) \ , \qquad \text{for some } (d-1)-\text{form} \quad Y(X)$$

$$\Downarrow$$

the ectoplasm action S is invariant under X-transformations and then full local supergravity gauge transformation $\mathcal{K}$

Conformal SUGRA actions: $\mathcal{N}$-extended CS term

- From idea of the component in 80s, the first natural object to start with is the Chern-Simon 3-form for $\mathrm{OSp}(\mathcal{N}|4,\mathbb{R})$
- denote the $\mathfrak{osp}(\mathcal{N}|4,\mathbb{R})$ generators collectively by $X_{\tilde{a}}$, $f_{\tilde{a}\tilde{b}}{}^{\tilde{c}}$ are the structure constants

$$[X_{\tilde{a}}, X_{\tilde{b}}\} = -f_{\tilde{a}\tilde{b}}{}^{\tilde{c}} X_{\tilde{c}} \ , \quad f_{\tilde{a}\tilde{b}}{}^{\tilde{c}} = -(-1)^{\varepsilon_{\tilde{a}}\varepsilon_{\tilde{b}}} f_{\tilde{b}\tilde{a}}{}^{\tilde{c}} \ .$$

- Cartan-Killing metric:

$$\Gamma_{\tilde{a}\tilde{b}} = f_{\tilde{a}\tilde{d}}{}^{\tilde{c}} f_{\tilde{b}\tilde{c}}{}^{\tilde{d}} (-1)^{\varepsilon_{\tilde{c}}} \ , \qquad \Gamma_{\tilde{a}\tilde{b}} = (-1)^{\varepsilon_{\tilde{a}}\varepsilon_{\tilde{b}}} \Gamma_{\tilde{b}\tilde{a}}$$

- With Cartan-Killing construct a gauge invariant closed four-form:

$$\langle R^2 \rangle := R^{\tilde{b}} \wedge R^{\tilde{a}} \Gamma_{\tilde{a}\tilde{b}} \ , \quad \mathrm{d}\langle R^2 \rangle = 0 \ .$$

- Chern-Simons three-form for $\mathfrak{osp}(\mathcal{N}|4,\mathbb{R})$

$$\Sigma_{\mathrm{CS}} = R^{\tilde{b}} \wedge \omega^{\tilde{a}} \Gamma_{\tilde{a}\tilde{b}} + \frac{1}{6} \omega^{\tilde{c}} \wedge \omega^{\tilde{b}} \wedge \omega^{\tilde{a}} f_{\tilde{a}\tilde{b}\tilde{c}} \ , \quad \mathrm{d}\Sigma_{\mathrm{CS}} = \langle R^2 \rangle \ .$$

not closed

Conformal SUGRA actions: $\mathcal{N}$-extended CS term and Σ_R

- Philosophy is then to search for a "curvature induced" 3-form Σ_R

$$\mathrm{d}\Sigma_R = \langle R^2 \rangle$$

constructed only out of the covariant fields of conformal superspace

- If Σ_R exists, then we have found a closed 3-form $\mathfrak{J}$

$$\mathfrak{J} = \Sigma_{\rm CS} - \Sigma_R \ , \quad \mathrm{d}\mathfrak{J} = 0$$

that gives the action of $\mathcal{N}$-extended conformal supergravity.

Note: for $\mathcal{N} = 1, 2$ $\langle R^2 \rangle = 0 \quad \Longrightarrow \quad \mathfrak{J} = \Sigma_{\rm CS}$
the CS form is closed and generate the conformal SUGRA action
No surprizes: $\mathcal{N} = 1, 2$ are generated by CS action for conformal group as [van Nieuwenhuizen (1985)]; [Roček, van Nieuwenhuizen (1986)]

$\mathcal{N}$-extended curvature induced form

- want Σ_R only constructed in terms of W^{IJKL}

the only possible ansatz for the lowest components of Σ_R is

$$\Sigma_{\alpha}^{I}{}_{\beta}^{J}{}_{\gamma}^{K} = 0 \ , \quad \Sigma_{a}{}_{\beta}^{J}{}_{\gamma}^{K} = \mathrm{i}\,(\gamma_a)_{\beta\gamma}\Big(A\,\delta^{JK}W^{ILPQ}W_{ILPQ} + B\,W^{LPQJ}W_{LPQ}{}^{K}\Big) \ ,$$

Plug in $\mathrm{d}\Sigma_R =< R^2 >$ at lowest mass dimension you get

$$0 \quad = \quad E_L^{\delta}E_K^{\gamma}E_J^{\beta}E_I^{\alpha}\,\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta}\Big(-W^{PQIJ}W^{KL}{}_{PQ} + A\,W^{PQRS}W_{PQRS}\delta^{J[K}\delta^{L]I} + B\,W^{PQRJ}W_{PQR}{}^{[K}\delta^{L]I}\Big)$$

first term contains a double traceless contribution of the form

$$\Big(\delta^R_{[K}\delta^{[I}_{|S|} - \frac{1}{\mathcal{N}}\delta^R_S\delta^{[I}_{[K}\Big)\Big(\delta^{|T|}_{L]}\delta^{J]}_U - \frac{1}{\mathcal{N}}\delta^{J]}_{L]}\delta^T_U\Big)W^{SUPQ}W_{RTPQ}$$

which cannot be cancelled by the Ansatz for $\mathcal{N} > 5$...
but $\mathcal{N} = 3, 4, 5$ work easily

Why for $\mathcal{N} > 5$ didn't work? example $\mathcal{N} = 6$

There is a very simple physical explanation: we are not considering all the relevant degrees of freedom in our construction

- the $\mathcal{N} = 6$ Weyl multiplet possesses the component field $X_{\alpha_1\alpha_2}{}^{I_1\cdots I_6} = X_{\alpha_1\alpha_2}\varepsilon^{I_1\cdots I_6}$ (for a U(1) potential A_a)

$$\partial^{\alpha_1\alpha_2}X_{\alpha_1\alpha_2} = 0 \ , \qquad X_{\alpha\beta} = (\gamma_a)_{\alpha\beta}\varepsilon^{abc}\partial_b A_c$$

$\mathcal{N} = 6$ Weyl multiplet has extra U(1) besides $\mathfrak{osp}(\mathcal{N}|4, \mathbb{R})$

$$\Downarrow$$

- U(1) super 2-form $F = \mathrm{d}A$ out of $W^{IJ} := (1/4!)\varepsilon^{IJKLPQ}W_{KLPQ}$
- Modify CS-form as

$$\Sigma_{\rm CS} = R^{\tilde{b}} \wedge \omega^{\tilde{a}}\Gamma_{\tilde{a}\tilde{b}} + \frac{1}{6}\omega^{\tilde{c}} \wedge \omega^{\tilde{b}} \wedge \omega^{\tilde{a}} f_{\tilde{a}\tilde{b}\tilde{c}} - aF \wedge A$$
$$\mathrm{d}\Sigma_{\rm CS} = \langle R^2 \rangle \ - aF \wedge F$$

- $\mathcal{N}$-extended Ansatz for Σ_R now works: $\Sigma_R \Longrightarrow \mathfrak{J}$, $\mathrm{d}\mathfrak{J} = 0$ exists!
- use ectoplasms and reduce to components: component action for $\mathcal{N} = 6$ conformal supergravity (same as Nishimura-Tanii)

$\mathcal{N} = 6$ conformal SUGRA action

The off-shell $\mathcal{N} = 6$ action in components is

$$S = \frac{1}{4}\int \mathrm{d}^3x\, e\,\Big\{\varepsilon^{abc}\big(\omega_a{}^{fg}\mathcal{R}_{bc\,fg} - \frac{2}{3}\omega_{af}{}^{g}\omega_{bg}{}^{h}\omega_{ch}{}^{f} - \frac{\mathrm{i}}{2}\Psi_{bc}{}_I^{\alpha}(\gamma_d)_\alpha{}^\beta(\gamma_a)_\beta{}^\gamma\varepsilon^{def}\Psi_{ef}{}_\gamma^I$$
$$- 2\mathcal{R}_{ab}{}^{IJ}V_{c\,IJ} - \frac{4}{3}V_a{}^{IJ}V_{bI}{}^{K}V_{c\,KJ} + 4\mathcal{F}_{ab}A_c\big)$$
$$- 4y^{IJ}w_{IJ} - \frac{16\mathrm{i}}{3}\tilde{w}^{\alpha\,IJK}\tilde{w}_{\alpha\,IJK} + 8\mathrm{i}X^\gamma_K X^K_\gamma - \frac{8}{3}\varepsilon^{IJKLPQ}w_{IJ}w_{KL}w_{PQ}$$
$$- 8\mathrm{i}\psi_a{}^\alpha_I(\gamma^a)_\alpha{}^\beta(\tilde{w}_\beta{}^{IJK}w_{JK} + X_{\beta J}w^{IJ})$$
$$+ 4\mathrm{i}\varepsilon^{abc}(\gamma_a)_{\alpha\beta}\psi_b{}^\alpha_I\psi_c{}^\beta_J(w^{IK}w^J{}_K - \frac{1}{4}\delta^{IJ}w^{KL}w_{KL})\Big\}$$

where the components are:

$$w^{IJ} := \frac{1}{4!}\varepsilon^{IJKLPQ}W_{KLPQ}|\ , \quad \tilde{w}_\alpha{}^{IJK} := -\frac{\mathrm{i}}{2}\nabla_\alpha^{[I}W^{JK]}|\ ,$$
$$y^{IJ} := -\frac{\mathrm{i}}{5}\nabla^{\gamma[I}\nabla_{\gamma P}W^{J]P}| - \frac{1}{2}\varepsilon^{IJKLPQ}W_{KL}W_{PQ}|\ , \quad X_\alpha{}^I := -\frac{\mathrm{i}}{5}\nabla_{\alpha J}W^{IJ}|$$

action for $\mathcal{N} = 3, 4, 5$ computed analogously
or by truncation of the $\mathcal{N} = 6$ action

Conclusion

- We constructed $\mathcal{N} = 1, 2, 3, 4, 5, 6$ conformal SUGRA action in 3D
- construction based on a new formulation of $\mathcal{N}$-extended conformal SUGRA in “conformal superspace” and “Ectoplasms”

Some open problems:

- Construction of non-linear $\mathcal{N} = 3, 4, 5, 6$ TMG. extention to GMG
- What about $\mathcal{N} > 6$?