Supersymmetric Field Theories on Curved Spaces

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UNIVERSITÀ DEGLI STUDI DI MILANO BICOCCA

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FUTURO IN RICERCA

Introduction

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Many recent examples:

- $\mathcal{N} = 2$ on S^4 [Pestun '07]
- $\mathcal{N} = 2$ on
  - round S^3 [Kapustin, Willett, Yaakov '09] [Jafferis'10]
  - squashed S^3 [Hama, Hosomichi, Lee '10] [Imamura, Yokoyama '11]

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In all these cases, curved geometry would break susy;
one has to deform the action by suitable terms.

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In this talk we will see, for theories with an R-symmetry:

- 4d Euclidean $\mathcal{N} = 1$ $\Leftarrow$ complex manifold

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- 4d Lorentzian $\mathcal{N} = 1$ $\Leftrightarrow$ manifold with a null conformal Killing vector (CKV)

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Similar results exist for 3d [Klare, AT, Zaffaroni '12] [Hristov, AT, Zaffaroni '13]

I will also sketch a holographic application to black holes

Plan

1. Strategy: coupling to supergravity
2. Classification theorems
3. Holographic applications

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fields $g_{\mu\nu}$, ψ_μ, A_μ

$$L_{\text{curved}} + L_{\text{sugra}}$$

$\partial_\mu \to \nabla_\mu$

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- If we find a configuration $\{g_{\mu\nu}, \psi_\mu, \text{aux. fields}\}$

  invariant under some susy δ_ϵ

  then L_{curved} is invariant under δ_ϵ

  and we can make the sugra fields non-dynamical

- set $\psi_\mu = 0$;

susy parameter

superconformal parameter

susy transformation: $\delta\psi_\mu = \nabla_\mu^A \epsilon + \gamma_\mu \eta = 0$

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(susy parameter: ϵ; superconformal parameter: η; $\nabla^A_\mu = \nabla_\mu - iA_\mu$)

$\gamma^\mu \delta\psi_\mu \;\Rightarrow\; \eta = -\frac{1}{4} D^A \epsilon$

(Dirac operator $\gamma^\nu \nabla^A_\nu$)

$$\Rightarrow \quad \left(\nabla^A_\mu - \frac{1}{4}\gamma_\mu D^A\right)\epsilon = 0$$

$$\left[\nabla^A_{\alpha(\dot\beta}\epsilon_{\dot\gamma)} = 0\right]$$

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- Natural equation in conformal geometry:

$$\nabla_\mu \epsilon \;\in \text{vector} \otimes \text{spinor}$$
$$= \text{spinor} \otimes \text{'gravitino'}$$

$D\epsilon$ $\qquad$ $\left(\nabla_\mu - \tfrac{1}{d}\gamma_\mu D\right)\epsilon$

- The same equation can be obtained from holography.

[Klare, AT, Zaffaroni, '12;
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can be obtained by gauge-fixing conformal supergravity.

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- What about non-conformal theories?

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The reason is basically that ordinary supergravity
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- We will now classify conformal Killing spinors.

$$\left(\nabla_\mu^A - \frac{1}{d}\gamma_\mu D^A\right)\epsilon = 0$$

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- For $A \neq 0$, we are on our own.

Let's assume ϵ is chiral (no loss of generality)

and that it has no zeros (unlike for S^4).

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symplectic form
[Kähler if closed]

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a.c.s. is integrable

|||

M_4 is complex

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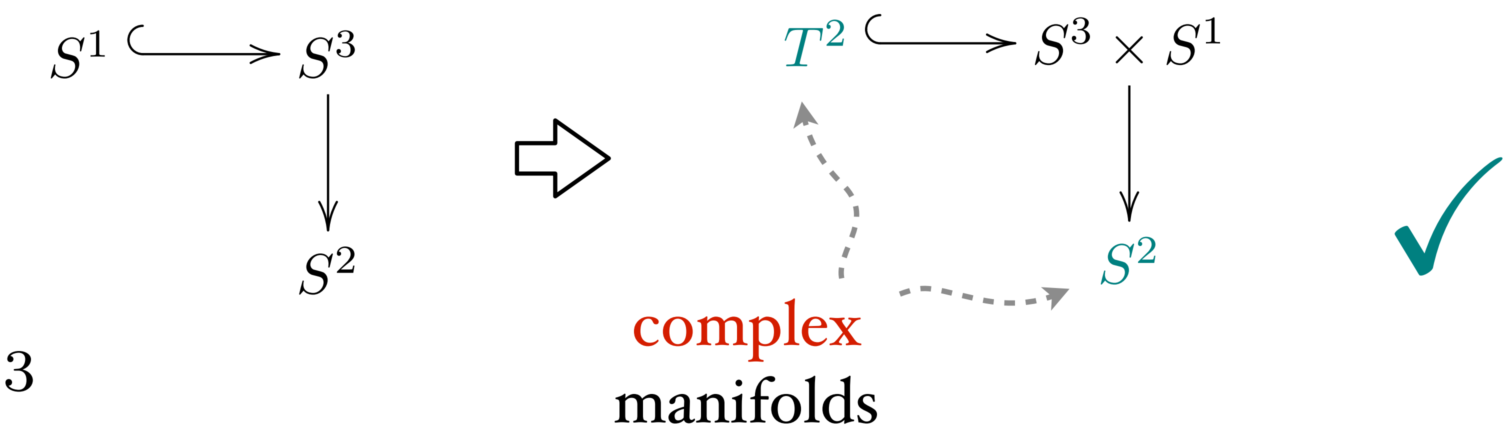

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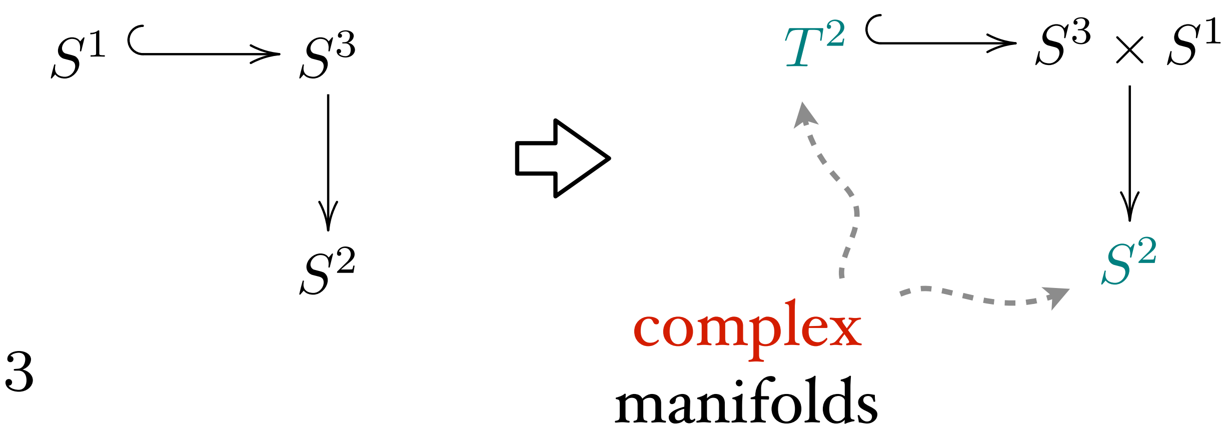

✓

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- Kähler manifolds

In this case holonomy is reduced to $\mathrm{U}(2) = \mathrm{U}(1) \times \mathrm{SU}(2)$

So $\exists$ A such that

$$\boxed{\nabla^A_\mu \epsilon_+ = 0}$$

which of course
is also a CKS

$$\left(\nabla^A_\mu - \tfrac{1}{4}\gamma_\mu D^A\right)\epsilon_+ = 0$$

This kind of 'trivial solution' to the CKS equation
is what we usually call a twist.

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  S^4 doesn't even admit an almost complex structure...

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  But recall: $\exists$ CKS with zeros

  $S^4 - \{\text{north pole}\} \cong \mathbb{R}^2$ does have a complex structure.

  ✓

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complex vector

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[Cassani, Klare, Martelli, AT, Zaffaroni '12]

$$\left(\nabla^A_\mu - \tfrac{1}{4}\gamma_\mu D^A\right)\epsilon_+ = 0 \iff \begin{cases} A = \dots \\ L_z g_{\mu\nu} = \alpha g_{\mu\nu} \end{cases}$$

[for some α]

z is a null CKV

conformal Killing vector

Similar results hold in $3\text{d}\,\mathcal{N}=2$

	$4\text{d}\,\mathcal{N}=1$	$3\text{d}\,\mathcal{N}=2$
Euclidean*	complex	o complex 1-form s.t. $do = w \wedge o$
Lorentzian	null CKV	null or timelike CKV

[Klare, AT, Zaffaroni '12]
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[Klare, AT, Zaffaroni '12]

[Cassani, Klare, Martelli, AT, Zaffaroni '12]

[Hristov, AT, Zaffaroni '13]

[* see also Komargodski's talk]

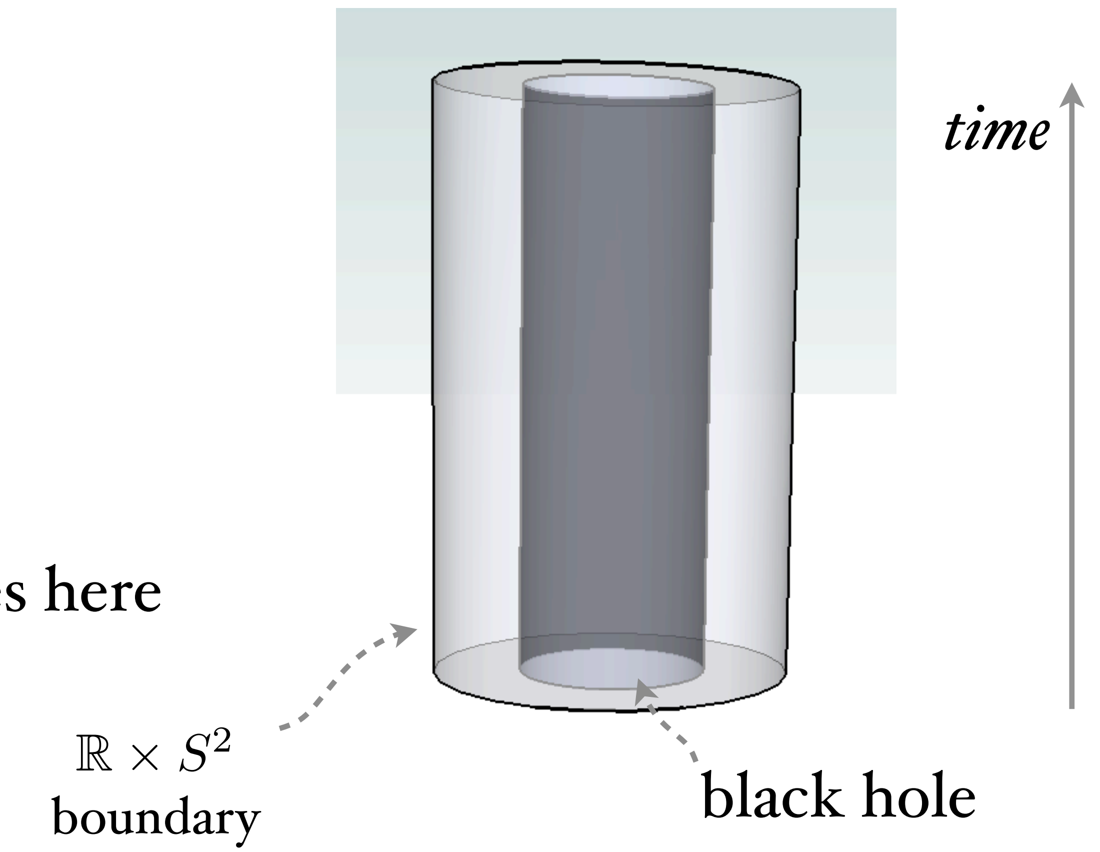
[Closset, Dumitrescu, Festuccia, Komargodski, '12]
[Closset, Dumitrescu, Festuccia, Komargodski, Shamir, to appear]

III. Holographic application

Asymptotically AdS4 black holes

$$ds_4^2 = \frac{dr^2}{r^2} + (r^2 ds^2_{\mathbb{R}\times S^2} + O(r))$$

boundary: a conformal theory lives here



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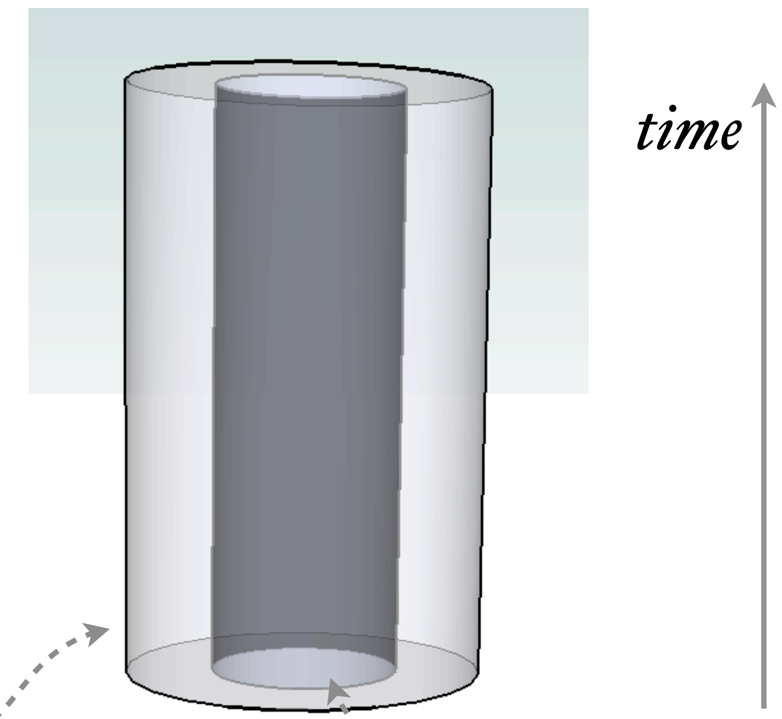

We classified all null or timelike CKVs on $\mathbb{R} \times S^2$: [Hristov, AT, Zaffaroni '13]

[$F = dA$ background field-strength]

$F = 0$

[$z = \partial_t + \partial_\phi$, for ex.]

$F = -\frac{1}{2}\mathrm{vol}_{S^2}$

$z = \partial_t$ 'twist'

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These two cases reproduce the two empirically known asymptotic behaviors of AdS4 BPS black holes!

[$F =$ graviphoton field-strength]

$$F = 0$$

1/2 BPS; naked sing.!

$$F = -\tfrac{1}{2}\mathrm{vol}_{S^2}$$

1/4 BPS, finite horizon

[interpolating family: rotating BHs]

[Cacciatori, Klemm '09]

In fact, the 1/4 BPS solutions look like

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this would give a way to count the entropy using the AdS2/CFT1 correspondence.

[Hristov, AT, Zaffaroni '13]

[Hristov, Rosa, AT, Zaffaroni, in progress!]

Conclusions

- A supersymmetric theory (with R-symmetry, low enough susy) is still supersymmetric on curved spaces if a CKS (or 'twistor') exists

- Existence of CKSs is equivalent to elegant geometrical properties

- These classification results restrict the possible asymptotic behaviors of AdS BPS black holes

Backup slides

- Actually, for higher susy a second (tougher) equation appears.

eg. for $d=4, \mathcal{N}=2$ / $d=6, \mathcal{N}=1$ also a $\delta\lambda$ appears.

[Klare, Zaffaroni '13]

e.g. [Samtleben,Sezgin,Tsimpis'13]

In this talk, we will keep susy low enough so that this complication does not appear:

$$d=4, \mathcal{N}=1$$

$$d=3, \mathcal{N}=2$$

line bundle

ϵ_+ is twisted by A $\Rightarrow$ section of $\Sigma_+ \otimes \mathcal{U}$

$\Rightarrow$ ω section of $K \otimes \mathcal{U}^2$ $\Rightarrow$ it can exist globally